A UNIVERSAL BOUND FOR THE AVERAGE COST OF ROOT FINDING

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ABSTRACT. We analyze a path-lifting algorithm for finding an approximate zero of a complex polynomial, and show that for any polynomial with distinct roots in the unit disk, the average number of iterates this algorithm requires is universally bounded by a constant times the log of the condition number. In particular, this bound is independent of the degree d of the polynomial. The average is taken over initial values z with $|z| = 1 + 1/d$ using uniform measure.

CONTENTS

1. INTRODUCTION

A point z_* is an ***approximate zero*** for a function $f(z)$ if it converges quadratically to a zero under Newton's method. The notion of an approximate zero was introduced by Smale in [Sm81]. A sufficient condition to determine if z_* is an approximate zero for f using only evaluation of $f(z_*)$ and its derivatives at z_* was developed by Kim ([K85], [K88]); this condition was sharpened and extended to apply also to systems of polynomials by Smale [Sm86]. Nowadays, this approach is commonly called α-theory. We will use the Kim-Smale criterion to locate approximate zeros; see Theorem 3.2.

Depending on the context of the problem, the goal might be to produce a point $\hat{z}$ so that $|f(\hat{z})| < \varepsilon$, or one might desire that $|\hat{z} - \zeta| < \varepsilon$ where $f(\zeta) = 0$. In either case, such a solution is called

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an ***ε*-root**. Notice that an approximate zero z_* may not be an ε-root. However, such a point z_* will converge to an ε-zero quadratically. Consequently, locating an approximate zero resolves the question of producing an ε-root. See Def. 3.1 for the specific definition.

In this paper, we discuss the use of a path-lifting method (which we call the ***α*-step method**, a variation of the algorithm developed in [K85] and [K88]) to locate an approximate zero for a complex polynomial $f(z)$, and show that for any polynomial f, the average number of steps required by the algorithm is universally bounded, independent of the degree of f, where the average is taken over the starting points for the method. In fact, the average cost depends on the average of the logarithms of several of the critical values of f (this, in turn, is less than the logarithm of the condition number; see Remark 3.8). We note that the results of [K88] imply that for any polynomial, this method converges except on a finite set of starting values.

More precisely, we have the following.

Theorem 1.1 (Main Theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial with distinct roots ζ_i in the unit disk. There is a constant Λ_f, independent of the degree of f, so that the average number of steps required by the α-step algorithm to locate an approximate zero for f is no more than*

$$67(\Lambda_f + 13.1),$$

where the average is taken over starting values on the circle of radius $1 + 1/d$. The constant Λ_f is the average of the logarithms of the radius of convergence of f^{-1} at the roots ζ_i.

The cost of each step of the algorithm is dominated by the calculation of $\alpha_f(z)$. Since this can be done with $\mathcal{O}(d\log^2 d)$ arithmetic operations (see [BM], for example), we have the following.

Corollary 1.2. *The average arithmetic complexity of locating an approximate zero for f by the α-step algorithm is $\mathcal{O}\left(\Lambda_f d\log^2 d\right)$, where the average is taken over starting values on the circle of radius $1 + 1/d$.*

In path-lifting methods, it is useful to distinguish between the domain and range, so we have

$$f : \mathbb{C}_{source} \to \mathbb{C}_{target}.$$

To implement the method, we choose a path γ in the target space (typically a segment connecting an initial point $w_0 = f(z_0)$ to zero) and attempt to lift it back to the source space via a branch of f^{-1}. In this form, such methods were introduced by Shub and Smale (see, for example [SS86] or [Sm85]), although one could argue (as Smale points out in [Sm81]) that in some sense this idea goes back to Gauss. See [Ren] and the references therein, as well as [KS]. The series [SS93a, SS93b, SS93c, SS96, SS94, Sh07, BS] discusses related methods for systems of polynomial equations. A survey of complexity results for solving polynomial equations in one variable can be found in [Pa]; see also [B08].

The difficulty of computing a local branch of f^{-1} along a path γ in the target space is related to how close γ comes to a critical value of f. However, not all critical values of f are relevant: only those which correspond to a critical point of f lying near the particular branch of $f^{-1}(\gamma)$ have any impact. Consequently, it is useful to factor f through the (branched) Riemann surface $\mathscr{S}$ for f^{-1},

so that we have

$$\begin{tikzcd} \mathbb{C}_{source} \arrow[r, "\widehat{f}"] \arrow[dr, "f"'] & \mathscr{S} \arrow[d, "\pi"] \\ & \mathbb{C}_{target} \end{tikzcd}$$

where $\widehat{f}$ is a diffeomorphism except at the critical points of f, and π is the projection map. With this viewpoint, the path γ that we lift back to $\mathbb{C}_{source}$ lies in $\mathscr{S}$, and the troublesome points now are the branch points of $\mathscr{S}$.

One ingredient important to our analysis is understanding the Voronoi decomposition of $\mathscr{S}$ relative to the branch points. That is, for each branch point v of $\mathscr{S}$, the Voronoi domain $\mathrm{Vor}(v)$ is the set of points in $\mathscr{S}$ which are closer to v than any other branch point of $\mathscr{S}$ (using the metric lifted via π). We show in §4 that the projection map π restricted to any single $\mathrm{Vor}(v)$ is at most $(m+1)$-to-one, where m is the multiplicity of the critical point of f corresponding to v (hence the projection is generically at most 2-to-one). Because the number of steps required for a path-lifting algorithm is related to the number of critical values the lifted path comes near, this result enables us to count the number of relevant critical values for a typical path.

The path-lifting algorithm works as follows: We choose an initial point z_0 just outside the disk known to contain all of the roots, and let $w_0 = f(z_0)$. We then attempt to continue the branch of f^{-1} which has $f^{-1}(w_0) = z_0$ along the segment from w_0 to 0 by choosing a suitable sequence w_n along this ray, together with approximations z_n such that $f(z_n) \approx w_n$. (Note that specifying a pair (z,w) such that $w = f(z)$ is equivalent to specifying a point in $\mathscr{S}$, so in practice our rays live naturally in $\mathscr{S}$.) The process stops when a point z_n is detected to be an approximate zero.

Given a pair (z_n, w_n), the point w_{n+1} is chosen to ensure that z_n is an approximate zero for $f(z) - w_{n+1}$. The tool we use to detect approximate zeros is the Kim-Smale α-function: if $\alpha_f(z) < 0.1307$, then z is an approximate zero.

The paper is organized as follows. In section 2, we set out notation and preliminary notions. Section 3 describes the path-lifting algorithm explicitly. In section 4, we discus the branched surface $\mathscr{S}$ and the corresponding Voronoi partition; this section may be of interest independent to the question of root-finding. Section 5 computes several estimates related to how the polynomial f behaves on the initial circle. In section 6, we calculate a lower bound on how far apart the points w_n and w_{n+1} can be, and in §7 bound the number of steps needed for the algorithm to locate an approximate zero from a given starting point z_0. This bound depends on the log of the angle $f(z_0)$ makes with the relevant critical values of f.

Using these results, we can average the bound from §7 over all starting points; this is done in Section 8, which proves the main theorem. We conclude in Section 9 with some remarks.

2. Preliminaries

We will use the following general notions and notations throughout.

An open disk of radius $r > 0$ centered around $z \in \mathbb{C}$ is denoted by $D_r(z)$.
The function Arg denotes the argument of a complex number (in the interval $(-\pi, \pi]$).

The ***ray*** $\ell_z \subset \mathbb{C}$ of a point $z \in \mathbb{C} \setminus \{0\}$ is

$$\ell_z = (0,\infty)\cdot z = \{w \in \mathbb{C} \mid \operatorname{Arg} w = \operatorname{Arg} z\},$$

and the ***slit*** of this point is the part of the ray extending outward from z, that is

$$\sigma_z = [1,\infty)\cdot z = \{w \in \ell_z \mid |w| \geq |z|\}$$

Finally, we will introduce some notation used when dealing with the Newton flow. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial, and denote the critical points of f by

$$\mathscr{C}_f = \left\{z \mid f'(z) = 0\right\}.$$

Consider the following vector field on $\mathbb{C}$,

$$X(z) = -\frac{f(z)}{f'(z)}.$$

The corresponding flow is called the ***Newton flow***. This vector field blows up near the critical points of f. By rescaling the length of the vector $X(z)$ by $|f'(z)|^2$, the critical points of f become well-defined singular points of the rescaled vector field. This rescaled vector field is the gradient vector field $\dot{z} = -\nabla|f(z)|^2$; the solution curves of the former coincide with the latter, and we will use the two interchangably. The equilibria of the Newton flow are exactly the roots and critical points of f. Each root ζ is a sink; we shall denote its basin of attraction by $\mathrm{Basin}(\zeta)$. Critical points are saddles for the flow. Furthermore, we can extend the flow to infinity, which is the only source. Each boundary component of $\mathrm{Basin}(\zeta)$ contains critical points $c \in \mathscr{C}_f$; each critical point c has an unstable orbit leaving from c and converging to ζ. This unstable orbit is a separatrix of c and will be denoted by γ_c. Generically, there is a unique critical point in each boundary component; in the degenerate cases, there could be saddle connections resulting in multiple critical points on one boundary component. A general discussion regarding Newton flows can be found in [STW] and [JJT]. See also Figure 4.2.

We note that for each root ζ, f is a biholomorphic map between $\mathrm{Basin}(\zeta)$ and $\mathbb{C} \setminus \bigcup \sigma_{f(c)}$, where the union is taken over the critical points c which lie on the boundary of $\mathrm{Basin}(\zeta)$.

It is important to note that if ϕ_t is a solution curve for the Newton flow, $f(\phi_t)$ lies along a ray.

Throughout the paper, we will consider polynomials $f \in \mathscr{P}_d(1)$, that is, $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \prod_{j=1}^{d} (z - \zeta_j) \qquad \text{with } |\zeta_j| < 1,$$

with distinct roots ζ_j. The set of roots of f will be denoted by

$$\mathscr{Z}_F = \left\{\zeta_j \mid j = 1,\ldots,d\right\}.$$

The restriction to $\mathscr{P}_d(1)$ is not severe; it can always be accomplished via an affine change of coordinates depending only on the coefficients of f; see [M], for example.

We shall use the following standard result several times.

Lemma 2.1. (Koebe Lemma) *Let $g : D_r(0) \to \mathbb{C}$ be univalent with $g(0) = 0$ and $g'(0) = 1$. For $z \in D_r(0)$ with $s = |z|/r$, we have*

$$\frac{1-s}{(1+s)^3} \leq |g'(z)| \leq \frac{1+s}{(1-s)^3} \tag{2.1}$$

and

$$|z|\frac{s}{(1+s)^2} \leq |g(z)| \leq |z|\frac{s}{(1-s)^2} \tag{2.2}$$

Consequently,

$$D_{r/4}(0) \subset g(D_r(0)). \tag{2.3}$$

Remark 2.2. The last statement (2.3) is known as the Koebe $\frac{1}{4}$-Lemma. The proof can be found in [Ko], [P].

3. The Path-Lifting Algorithm

We now discuss explicitly the path-lifting algorithm that we will use to find an approximate zero of an $f \in \mathscr{P}_d(1)$.

Definition 3.1. Let $z_n \in \mathbb{C}$ be the n^{th} iterate under Newton's method of the point $z_0 \in \mathbb{C}$, that is,

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}.$$

The point z_0 is called an ***approximate zero*** of f if

$$|z_{n+1} - z_n| \leq \left(\frac{1}{2}\right)^{2^n - 1} |z_1 - z_0|$$

for all $n > 0$.

A sufficient condition for a point to be an approximate zero is developed in [K85] and [Sm86]. We will use the criterion formulated by Smale in [Sm86] to locate approximate zeros. It uses $\alpha : \mathbb{C} \setminus \mathscr{C}_f \to \mathbb{C}$ defined by

$$\alpha(z) = \max_{j>1} \left|\frac{f(z)}{f'(z)}\right| \left|\frac{f^{(j)}(z)}{j!\, f'(z)}\right|^{\frac{1}{j-1}}. \tag{3.1}$$

It is sometimes useful to use the related function $\gamma(z)$ instead, where

$$\gamma(z) = \max_{j>1} \left|\frac{f^{(j)}(z)}{j!\, f'(z)}\right|^{\frac{1}{j-1}}. \tag{3.2}$$

While we will primarily use $\alpha(z)$, we make use of $\gamma(z)$ in section 6.

Theorem 3.2. [Sm86] *There is a number $\alpha_0 > 0.1307$ such that if $\alpha(z) < \alpha_0$, the point z is an approximate zero.*

Remark 3.3. The number α_0 is given in [Sm86] and in many places throughout the literature as $\alpha_0 \approx 0.130707$. However, this specific value is very likely the result of a typographic error in the fifth decimal place. Smale shows in [Sm86] that α_0 is a solution to the equation $(2r^2 - 4r + 1)^2 - 2r = 0$; the relevant root of this equation is $0.13071694\ldots$. There have been subsequent improvements to this constant (see [WH] or [WZ], for example), but 0.1307 suffices for our purposes.

Remark 3.4. Calculation of $\alpha(z)$ requires the ability to evaluate all derivatives of f at a z. In some situations, this is not possible; for example, if f is defined as an n-fold composition of some other function g, calculation of f and f' in terms of g and g' is simple, but calculating even f'' is impractical. However, evaluation of higher derivatives may be avoided using the bound [Sm86]:

$$\gamma(z) < \frac{\|f\|\,(\varphi_d'(|z|))^2}{|f'(z)| \cdot \varphi_d(|z|)}\,,$$

where $\varphi_d(z) = \sum z^i$, $f(z) = \sum a_i z^i$, and $\|f\| = \max |a_i|$. Alternatively, an adaptive version of the algorithm can be used which doesn't use α: see remark 2 in §9.

We shall analyze the following algorithm to find an approximate zero for $f \in \mathscr{P}_d(1)$.

The α-Step Path Lifting Algorithm

Step 0: Choose $z_0 \in \mathbb{C}$ with $|z_0| = 1 + \frac{1}{d}$. Let

$$w_0 = f(z_0) \quad \text{and} \quad w = \frac{w_0}{|w_0|}.$$

Step 1: Stop if $\alpha(z_n) \leq 0.1307$; z_n is an approximate zero for f.

Step 2: Let

$$w_{n+1} = w_n - \frac{1}{15} \cdot \frac{|f(z_n)|}{\alpha(z_n)} \cdot w$$

and

$$z_{n+1} = z_n - \frac{w_{n+1} - f(z_n)}{f'(z_n)}.$$

Continue with Step 1.

We shall sometimes refer to the points w_n generated by the algorithm above as ***guide points***.

Remark 3.5. If $z_0 \in \mathrm{Basin}(\zeta)$ then the algorithm will terminate with an approximate zero for ζ. There may be some values of n for which $z_n \notin \mathrm{Basin}(\zeta)$. However, there exists a neighborhood $U \subset \mathbb{C}$ of the ray of w_0 which contains all $f(z_n)$ and on which there exists a univalent inverse branch of f mapping w_0 to z_0. Denote this inverse branch by $f_{z_0}^{-1} : U \to \mathbb{C}$. See Figure 3.1.

For every zero $\zeta \in \mathscr{Z}_f$ consider the ***closest critical value to*** $0 = f(\zeta)$

$$\rho_\zeta = \min_{c \in \mathscr{C}_f(\zeta)} |f(c)| \qquad \text{where} \qquad \mathscr{C}_f(\zeta) = \mathscr{C}_f \cap \overline{\mathrm{Basin}(\zeta)}.$$

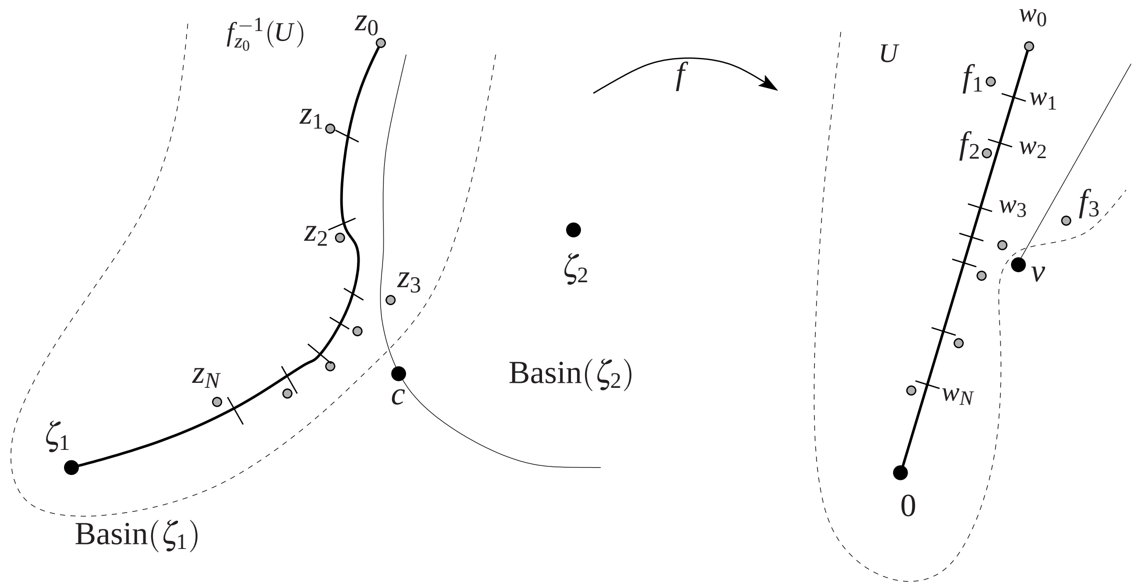

FIGURE 3.1. While $z_0 \in \text{Basin}(\zeta_1)$, we have $z_3 \in \text{Basin}(\zeta_2)$. However, as noted in remark 3.5, there is a neighborhood U of the ray on which the same branch of the inverse contains all the z_n.

Remark 3.6. Note that ρ_ζ is the radius of convergence of f^{-1} at ζ, and is the distance in the surface $\mathscr{S}$ between $\widehat{f}(\zeta)$ and the nearest branch point of $\mathscr{S}$.

For any polynomial f, we define

$$K_f = \prod_{\zeta \in \mathscr{Z}_f} \frac{1}{\rho_\zeta}.$$

Remark 3.7. Notice that $K_f < \infty$ if and only if $\mathscr{Z}_f \cap \mathscr{C}_f = \emptyset$. This holds generically for polynomials f, and $K_f = \infty$ exactly when f has a multiple zero. Root-finding problems for which there is a multiple zero are typically called ***ill-conditioned*** or ***ill-posed***.

Remark 3.8. K_f is related to the condition number μ_f, which is the reciprocal of the distance between f and an ill-conditioned problem (see [SS93b] and [Dem], for example).

It is common to use $\mu_f(\zeta)$ to denote the condition number considering only those problems for which ζ is a root. Note that

$$\mu_f(\zeta) \geq \frac{1}{\rho_\zeta} \qquad \text{and} \qquad \mu_f \geq \min_{\zeta \in \mathscr{Z}_f} \frac{1}{\rho_\zeta},$$

since the map $f(z) - v_\zeta$ has a multiple root at ζ, where v_ζ is the appropriate critical value with $|v_\zeta| = \rho_\zeta$.

Let $\#_f(z)$ be the number of steps the α-step algorithm needs to get to an approximate zero when it begins at z, which we refer to as the ***cost of the algorithm at*** z. The average number of steps is

denoted by

$$\overline{\#_f} = \int_0^1 \#_f\left(\left(1+\frac{1}{d}\right)e^{2\pi it}\right)dt,$$

where we use starting points on the circle of radius $1+1/d$.

Our main theorem (Thm. 1.1) says that if $f \in \mathscr{P}_d(1)$, we have

$$\overline{\#_f} \leq 67\left(13.1+\frac{2|\log K_f|}{d}\right).$$

The proof of the theorem will be prepared in the next sections, and is completed in section 8.

Remark 3.9. We analyze the algorithm using starting points $z \in \mathbb{C}$ with $|z| = 1+C/d$. Taking $C=1$ yields the bound stated above.

Remark 3.10. One can introduce a measure of difficulty $K_{f,\zeta}$ corresponding to a given zero $\zeta \in \mathscr{Z}_f$. Proposition 7.1 gives an estimate for the time needed to reach an approximate zero for ζ starting in z_0 in terms of $K_{f,\zeta}$.

4. The Voronoi Partition in the Branched Cover

Given a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree d, recall that we denote its critical points by $\mathscr{C}_f = \{z \mid f'(z) = 0\}$. For any such f, we can express it as a composition $f = \pi \circ \widehat{f}$,

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\widehat{f}"] \arrow[dr, "f"'] & \mathscr{S} \arrow[d, "\pi"] \\ & \mathbb{C} \end{tikzcd},$$

where $\widehat{f}$ is a diffeomorphism except on $\mathscr{C}_f$ (where it is merely a bijection), and π is a d-fold branched cover, ramified at points of $\mathscr{C}_f$. The metric on $\mathscr{S}$, denoted by dist, is such that π is a local isometry away from points in $\mathscr{V}_f = \widehat{f}(\mathscr{C}_f)$.

The ***multiplicity*** of a critical point $c \in \mathscr{C}_f$ is

$$m_c = \min\left\{k \mid f^{(k+1)}(c) \neq 0\right\}.$$

Notice that

$$\sum_{c\in\mathscr{C}_f} m_c = d-1.$$

The points in $\mathscr{V}_f$ are called ***critical values*** in $\mathscr{S}$, and we define the multiplicity m_v of $v = \widehat{f}(c) \in \mathscr{V}_f$ to be the multiplicity of c.

We note that for each root $\zeta \in \mathscr{Z}_f$,

$$\pi : \widehat{f}(\mathrm{Basin}(\zeta)) \to \mathbb{C} \setminus \bigcup_{v\in\mathscr{V}_f(\zeta)} \sigma_v$$

is an isometry (where $\mathscr{V}_f(\zeta) = f(\mathscr{C}_f(\zeta))$).

The ***Voronoi domain*** of a point $v \in \mathscr{V}_f$ is

$$\mathrm{Vor}(v) = \{u \in \mathscr{S} \mid \mathrm{dist}(u,v) \le \mathrm{dist}(u,v'), \forall v' \in \mathscr{V}_f\}.$$

Recall that an open disk of radius $r > 0$ around $u \in \mathscr{S}$ is denoted by $D_r(u)$.

Lemma 4.1. *$u \in \mathrm{Vor}(v)$ if and only if $\pi : D_{|u-v|}(u) \to D_{|u-v|}(\pi(u))$ is an isometry. In particular, if $u \in \mathrm{Vor}(v)$ then*

$$D_{|u-v|}(u) \cap \mathscr{V}_f = \emptyset.$$

Proof. If $u \in \mathrm{Vor}(v)$ then $D_{|u-v|}(u) \cap \mathscr{V}_f = \emptyset$. Thus, π is a local isometry on all of $D_{|u-v|}(u)$, and in particular, π is a global isometry on this disk. Conversely, If π is an isometry on all of $D_{|u-v|}(u)$, there can be no critical values in the disk, and so $u \in \mathrm{Vor}(v)$. □

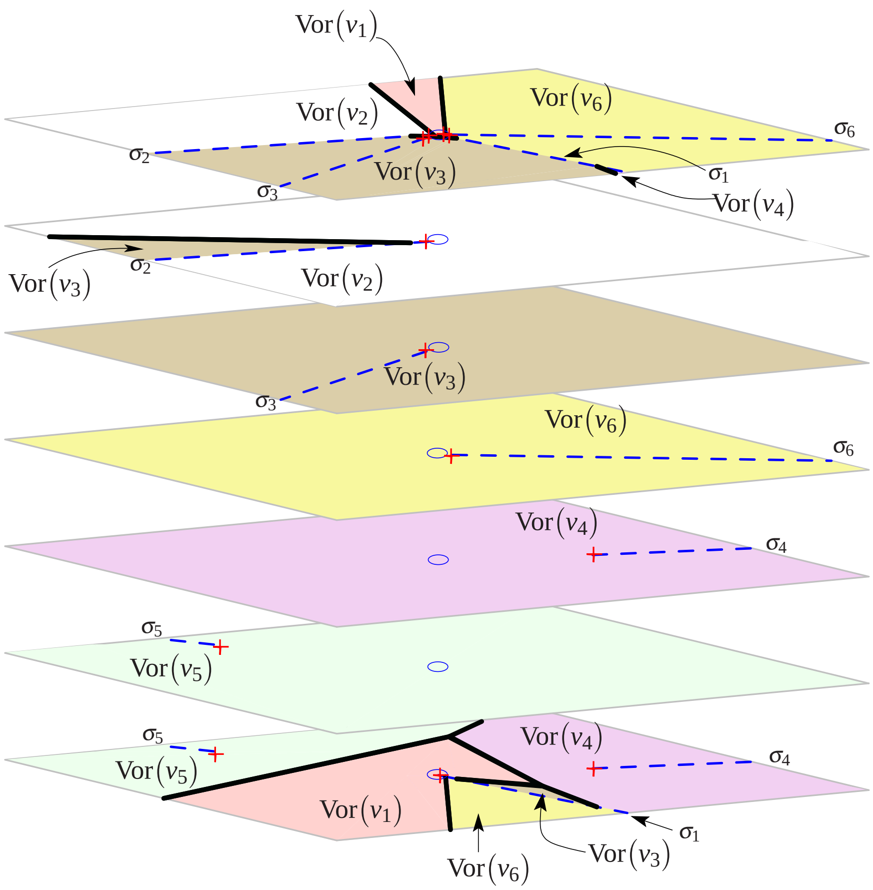

FIGURE 4.1. The surface $\mathscr{S}$ for a degree 7 polynomial, viewed as a stack of seven slit planes. Each sheet is $\widehat{f}(\mathrm{Basin}(\zeta_i))$ for the root ζ_i, and is slit along σ_{v_j} (dashed lines), which begin at the branch points $v_j \in \mathscr{V}_f$ (indicated by crosses). The central ellipses indicate $\pi^{-1}(0)$. In the figure, σ_{v_j} is labeled as σ_j. The Voronoi domains of each of the v_j are indicated, with boundaries marked by heavy black lines. Note that while $\mathrm{Vor}(v_j)$ may enter many sheets, the projection is at most 2-to-1, as in Cor. 4.5. See also Figure 4.2.

Let $u_1, u_2 \in \mathscr{S}$. If the line segment $[\pi(u_1), \pi(u_2)] \subset \mathbb{C}$ has a lift in $\mathscr{S}$ which connects u_1 with u_2, we denote this lifted line segment by $[\![u_1, u_2]\!]$. Observe that many pairs u_1, u_2 do not have such a connecting line segment. In this case we write $[\![u_1, u_2]\!] = \emptyset$. When $[\![u_1, u_2]\!]$ is nonempty, we say that u_1 ***is visible from*** u_2 in $\mathscr{S}$. Also observe, if $v \in \mathscr{V}_f$ then

$$[\![u, v]\!] \neq \emptyset \quad \text{for all } u \in \operatorname{Vor}(v).$$

We can form the ***visibility graph for*** $\mathscr{S}$ as follows. The vertices of the graph are the critical values $\mathscr{V}_f$, and there is an edge from v to w if and only if $[\![v, w]\!]$ is non-empty. We can identify the visibility graph with the subset of $\mathscr{S}$ given by

$$\mathscr{G} = \bigcup_{v, w \in \mathscr{V}} [\![v, w]\!].$$

Recall that $\widehat{f}$ is a homeomorphism. Hence, $\widehat{f}^{-1}(\mathscr{G})$ is well-defined, so we can also view $\mathscr{G}$ as a graph immersed in $\mathbb{C}$, with the critical points of f as vertices.

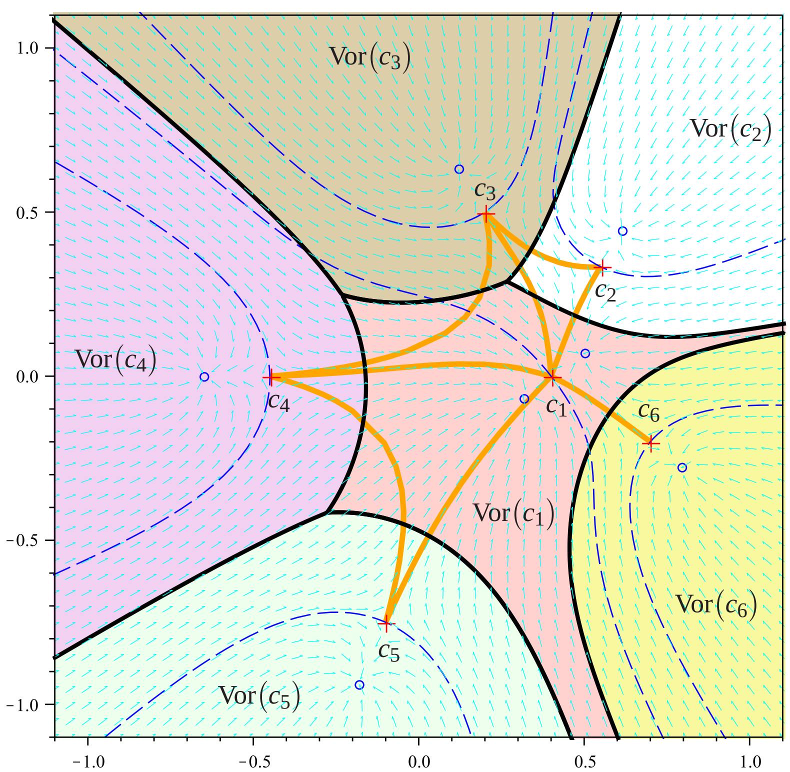

FIGURE 4.2. The Voronoi regions of Fig. 4.1 are shown in the source space. The roots of f are indicated by circles, the critical points by crosses. The Newton flow is indicated by the small arrows, and the dashed lines are the boundaries of the basins of each root (each such boundary contains a unique critical point). $\widehat{f}^{-1}(\operatorname{Vor}(v_j))$ is shown for each critical point $c_j \in \mathscr{C}_f$ (bounded by the heavy lines). In the figure, $\widehat{f}^{-1}(\operatorname{Vor}(v_j))$ is labeled by $\operatorname{Vor}(c_j)$. The visibility graph $\mathscr{G}$ is also shown.

We shall say that a critical value $w \in \mathscr{V}$ is a ***neighbor of*** v if there is an edge between w and v in $\mathscr{G}$. Denote the set of neighbors of v by $\mathscr{N}_v$.

For each edge of $\mathscr{G}$, we can define the lines

$$L_{v,w} = \{u \in \mathscr{S} \mid \operatorname{dist}(u,w) = \operatorname{dist}(u,v)\},$$

which are the geodesics passing perpendicularly through the midpoint of each $[\![v,w]\!]$. In the terminology of [VPRS] and [BV], each of the $L_{v,w}$ is a ***mediatrix*** relative to the set $\mathscr{V}_f$.

Problem 4.2. Which abstract graphs can be realized as the visibility graph $\mathscr{G}$ of a polynomial?

Lemma 4.3. *For $u \in \operatorname{Vor}(v)$ and $w \in \mathscr{N}_v$*

$$\left([\![u,v]\!] \smallsetminus \{u\}\right) \cap L_{v,w} = \emptyset.$$

Proof. According to Lemma 4.1, the metric on $D_{|u-v|}(u) \subset \mathscr{S}$ is the usual Euclidean metric. This implies immediately that if

$$[\![u,v]\!] \cap L_{v,w} \neq \emptyset$$

then either $w \in D_{|u-v|}(u)$ or $u \in L_{v,w}$; see Figure 4.3. If $w \in D_{|u-v|}(u)$, it cannot be in $\mathscr{V}_f$. □

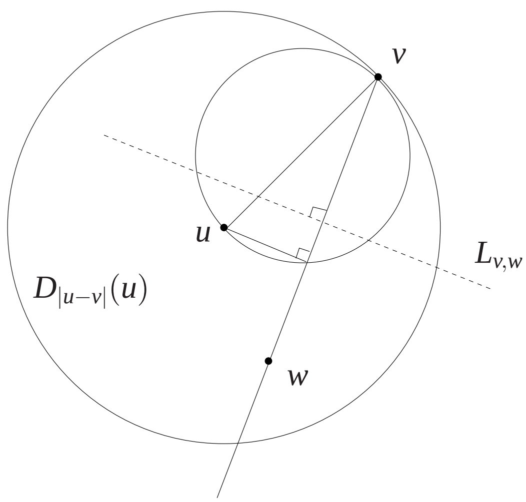

FIGURE 4.3. By Lemma 4.3, if $u \in \operatorname{Vor}(v)$, then $[\![u,v]\!]$ cannot cross $L_{v,w}$, since π is univalent on $D_{|u-v|}(u)$.

Lemma 4.3 can be used to describe the boundary of Voronoi domains. Specifically, for each $v \in \mathscr{V}_f$, $\operatorname{Vor}(v)$ is the connected component of

$$\mathscr{S} \smallsetminus \bigcup_{w \in \mathscr{N}_v} L_{v,w}$$

which contains v. See Figures 4.1 and 4.2.

Recall that the ray $\ell_y \subset \mathbb{C}$ of a point $y \in \mathbb{C} \smallsetminus \{0\}$ is the set of points which have the same argument as y.

If $\widehat{0} \in \mathscr{S}$ projects onto 0 and $[\![\widehat{0}, u]\!] \neq \emptyset$, the geodesic starting at $\widehat{0}$ and containing $[\![\widehat{0}, u]\!]$ is the ray through $u \in \mathscr{S}$, which we denote by $\widehat{\ell}_u$. Observe that if $\widehat{\ell}_u \cap \mathscr{V}_f = \emptyset$ then $\pi : \widehat{\ell}_u \to \ell_{\pi(u)}$, is a surjective isometry.

Let $y = \pi(u)$. If $\ell_y \cap f(\mathscr{C}_f) = \emptyset$, then

$$\pi^{-1}(\ell_y) = \widehat{\ell}_{y_1} \cup \widehat{\ell}_{y_2} \cup \cdots \cup \widehat{\ell}_{y_d},$$

where the points $y_i \in \mathscr{S}$ are the d different preimages of y.

Proposition 4.4. *Given $v \in \mathscr{V}_f$ and $y \in \mathbb{C} \setminus f(\mathscr{C}_f)$. Then*

$$\operatorname{card}\left\{ i \mid \widehat{\ell}_{y_i} \cap \operatorname{Vor}(v) \neq \emptyset \right\} \leq m_v + 1.$$

Furthermore, each $\widehat{\ell}_{y_i} \cap \operatorname{Vor}(v)$ is a connected set.

Proof. Suppose $\widehat{\ell}_{y_1}, \widehat{\ell}_{y_2}, \ldots, \widehat{\ell}_{y_k}$ intersect $\operatorname{Vor}(v)$, with $v = \widehat{f}(c)$, $c \in \mathscr{C}_f$. Pick a point u_i in each of these intersections, that is,

$$u_i \in \widehat{\ell}_{y_i} \cap \operatorname{Vor}(v).$$

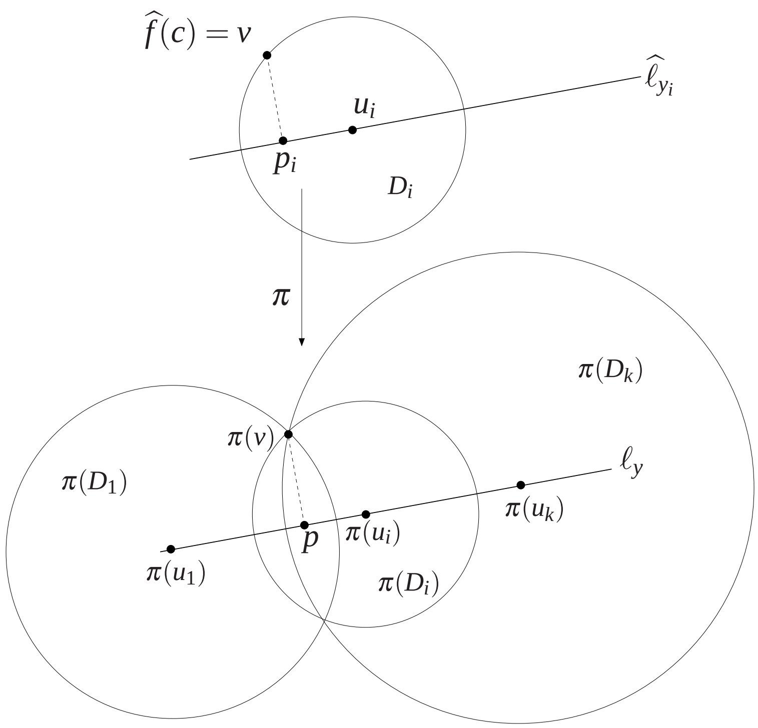

FIGURE 4.4. As proven in Proposition 4.4, the projection π is $(m_v + 1)$-to-one on $\operatorname{Vor}(v)$.

Let $D_i = D_{|v-u_i|}(u_i)$. According to Lemma 4.1, we know that $\pi : D_i \to \pi(D_i)$ is an isometry. Let $p_i \in \widehat{\ell}_{y_i}$ be the perpendicular projection of v onto $\widehat{\ell}_{y_i}$ and let p be the projection of $f(c) = \pi(v)$

onto ℓ_y. Then for all $i \le k$,

$$\emptyset \neq [\![v, p_i]\!] \subset D_i \qquad \text{and} \qquad \emptyset \neq [\pi(v), p] \subset \bigcap_{i \le k} \pi(D_i).$$

Hence, $\pi : \mathscr{S} \to \mathbb{C}$ is k-to-1 in a neighborhood of $v \in \mathscr{S}$, with $k \le m_c + 1$.

The connectedness of $\widehat{\ell}_{y_i} \cap \mathrm{Vor}(v)$ follows from the triangle inequality. □

Corollary 4.5. *Each projection* $\pi : \mathrm{Vor}(v) \to \mathbb{C}$ *is at most* $(m_v + 1)$*-to-one.*

Let $z \in \mathbb{C}$. We'll say that a critical point $c \in \mathscr{C}_f$ ***influences the orbit of*** z if the segment $[\![\widehat{0}, \widehat{f}(z)]\!]$ passes through $\mathrm{Vor}(\widehat{f}(c))$.

We are interested in the critical points which influence the starting values for our algorithm, and, conversely, the starting values which are influenced by a given critical point.

Definition 4.6. For starting values z on the circle of radius r, we define the following sets:

$$\mathscr{I} = \left\{ (t, c) \in [0,1] \times \mathscr{C}_f \mid [\![\widehat{0}, \widehat{f}(re^{2\pi i t})]\!] \cap \mathrm{Vor}\big(\widehat{f}(c)\big) \neq \emptyset \right\}$$

$$\mathscr{I}_t = \left\{ c \in \mathscr{C}_f \mid (t,c) \in \mathscr{I} \right\} \qquad\qquad \mathscr{I}_c = \{ t \in [0,1] \mid (t,c) \in \mathscr{I} \}$$

Notice that, for $z = re^{2\pi i t}$ fixed, we have $c \in \mathscr{I}_t$ precisely when, for some $y \in \ell_{f(z)}$, $D_{|f(c)-y|}(y)$ is the largest ball on which f_z^{-1} is defined. Similarly, for this pair (t, c), we also have $t \in \mathscr{I}_c$.

5. The Behavior of f on the Initial Circle

Consider the function $a_r : [0,1) \to \mathbb{R}$ defined by

$$a_r(t) = \mathrm{Arg}\, f(re^{2\pi i t}),$$

with $r > 0$. We can easily bound the rate of change of $a_r(t)$; while elementary, these bounds play a crucial role for us.

Lemma 5.1. *Let* $r > 1$*. Then for all* $t \in [0,1)$*, we have*

$$2\pi d \cdot \frac{r}{r+1} \le \frac{da_r}{dt} \le 2\pi d \cdot \frac{r}{r-1}.$$

Proof. Let $z = re^{2\pi i t}$, with $r > 1$. Since $|\zeta| < 1$, we have $\frac{\zeta}{z} \in D_{\frac{1}{r}}(0) = \left\{ w \mid |w| \le \frac{1}{r} \right\}$. A calculation shows

$$\frac{da_r}{dt} = \mathrm{Im} \frac{d}{dt} \log f(re^{2\pi i t}) = 2\pi \cdot \mathrm{Re} \sum_{j=1}^{d} \frac{z}{z - \zeta_j} = 2\pi \cdot \mathrm{Re} \sum_{j=1}^{d} \frac{1}{1 - \zeta_j / z}. \tag{5.1}$$

For each root ζ_i, we have

$$\frac{r}{r+1} \le \mathrm{Re} \frac{1}{1 - \zeta_i / z} \le \frac{r}{r-1}.$$

Summing this inequality over the d roots and applying it to equation 5.1 gives the desired result. □

Remark 5.2. The estimates in Lemma 5.1 are sharp.

Corollary 5.3. *Let $r = 1 + 1/d$, and define*

$$B_A = \left\{ t \in [0,1) \mid \left|\operatorname{Arg} \frac{\widehat{f}(re^{it})}{\widehat{f}(c)}\right| < A,\ \textit{for all } c \in \mathscr{C}_f \right\}.$$

Then

$$\textit{measure}(B_A) \le \frac{2A}{\pi} \cdot \frac{d-1}{d}.$$

Remark 5.4. Let G_A be the complement of B_A. For each $t \in G_A$, $f_{re^{it}}^{-1}$ will be analytic in a cone

$$\left\{ w \mid |\operatorname{Arg}(w) - \operatorname{Arg}(f(re^{it}))| < A \right\},$$

and consequently such t correspond to ”good starting points” for a path-lifting algorithm.

This is essentially Condition H of [Sm85] and [SS86], with $A = \pi/12$. It is shown in those papers (Prop. 2) that $G_{\pi/12}$ has measure at least $1/6$ if one also takes $r = 3/2$ (which increases the number of steps by approximately $d \log(3/2)$). The corollary 5.3 gives the measure of $G_{\pi/12}$ to be at least $5/6$.

Lemma 5.5. *Let c be a critical point on the boundary of* $\operatorname{Basin}(\zeta)$*, and let γ_c be the solution to the Newton flow emanating from c whose interior lies in* $\operatorname{Basin}(\zeta)$*. Then if $r > 1$, $\gamma_c \cap S_r = \emptyset$.*

Proof. Note that the Newton flow points inward on $S_r = \{z \mid |z| = r\}$ for $r > 1$, which follows from the observation that

$$\frac{f(z)}{f'(z)} = \frac{1}{\sum \frac{1}{z-\zeta_i}}.$$

This immediately implies Lemma 5.5.

To see this, note that since $|z| > 1$ and $|\zeta_i| \le 1$, the vectors $z - \zeta_i$ all lie in the half-plane $\mathscr{H}$ which contains D_r. Consequently, their inverses and hence their sum $\sum 1/(z-\zeta_i)$ lie in the complementary half-plane. Inverting again gives $f(z)/f'(z) \in \mathscr{H}$, as required. □

We can now use the previous lemmas to estimate the width of the “necks” of $\operatorname{Basin}(\zeta)$.

Lemma 5.6. *Let $r > 1$, $\zeta \in \mathscr{Z}_f$, and let γ be a connected component of $S_r \cap \overline{\operatorname{Basin}(\zeta)}$. Then*

$$\operatorname{length}(\gamma) \cdot \frac{1}{2\pi} \cdot \min \frac{da_r}{dt} \le 2\pi r.$$

Proof. Let $B \subset \overline{\operatorname{Basin}(\zeta)}$ be a boundary component of $\operatorname{Basin}(\zeta)$ with $\gamma \cap B \ne \emptyset$. Let $c \in \mathscr{C}_f \cap B$ be the critical point which has an orbit $\gamma_c \subset \overline{\operatorname{Basin}(\zeta)}$ of the Newton flow starting at c and ending at ζ.

Observe that

$$f(\gamma_c \cup B) = (0,\infty) \cdot f(c) = \ell_{f(c)},$$

the ray through $f(c)$. From the definition of γ and Lemma 5.5 we get $\operatorname{int}(\gamma) \cap (B \cup \gamma_c) = \emptyset$. Hence,

$$\operatorname{Arg}(f(\operatorname{int}(\gamma))) \cap \operatorname{Arg}(f(c)) = \emptyset,$$

that is, the image of γ cannot make more than a full turn in the target space. The Lemma follows. □

The following corollary follows immediately from the proof.

Corollary 5.7. *Let z_1 and z_2 satisfy $|z_1| = |z_2| = r$ with $r > 1$, and suppose also that they lie in the same connected component of $S_r \cap \mathrm{Basin}(\zeta)$. Then*

$$|\operatorname{Arg} f(z_1) - \operatorname{Arg} f(z_2)| < 2\pi.$$

In the sequel we will consider integrals over the circle $S_r = \{z \in \mathbb{C} \mid |z| = r\}$, which, for all $r > 0$, carries Lebesgue measure with unit mass.

Lemma 5.8. *Let $r > 0$ and $|\zeta| < r$ then*

$$\int_0^1 \log|re^{2\pi it} - \zeta|dt = \log r.$$

Proof. Define

$$\begin{aligned} S(\zeta) &= \int_0^1 \log|re^{2\pi it} - \zeta|dt \\ &= \int_{S_r} \mathrm{Re}(\log(z-\zeta)) \cdot \frac{1}{2\pi i}\frac{dz}{z} \\ &= \mathrm{Re}\,\frac{1}{2\pi i}\int_{S_r} \log(z-\zeta)\cdot\frac{dz}{z}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{dS}{d\zeta} &= -\mathrm{Re}\,\frac{1}{2\pi i}\int_{S_r}\frac{1}{z-\zeta}\frac{dz}{z} \\ &= -\mathrm{Re}\,\frac{1}{2\pi i}\int_{S_r}\left(\frac{1/\zeta}{z-\zeta} - \frac{1/\zeta}{z}\right)dz = 0. \end{aligned}$$

Hence,

$$S(\zeta) = S(0) = \log r.$$

□

Corollary 5.9. *Let $f(z) = \prod_{j=1}^d (z - \zeta_j)$, with $|\zeta_j| < r$. Then*

$$\int_0^1 \log|f(re^{2\pi it})|dt = d\log r.$$

Proof.

$$\begin{aligned} \int_0^1 \log|f(re^{2\pi it})|dt &= \int_0^1 \log\left|\prod_{j=1}^d (re^{2\pi it} - \zeta_j)\right| dt \\ &= \sum_{j=1}^d \int_0^1 \log|re^{2\pi it} - \zeta_j|dt = d\log r, \end{aligned}$$

where the last equality follows from Lemma 5.8. □

Remark 5.10. Notice that if $r = 1 + 1/d$, we have $d\log r < 1$.

Problem 5.11. The previous corollary shows that the average value of $\log|f(z)|$ on S_r is $d\log r$. Is there a constant c_r independent of d so that

$$\text{measure}\left\{t \mid \log|f(re^{2\pi it})| < d\log r\right\} > c_r?$$

We now establish a lower bound on $|w_0| = |f(z_0)|$ for starting values z_0 on the circle S_r with $r > 1$. We shall use this in Lemma 6.9 to give a bound on the size of our final point w_N.

Proposition 5.12. *Let $z \in \text{Basin}(\zeta)$ with $|z| = r > 1$. There exists $s_r < 1$ such that*

$$|f(z)| \geq s_r \cdot \rho_\zeta,$$

where ρ_ζ is the radius of convergence of the branch of f^{-1} taking 0 to ζ.

If $r > 1 + \frac{2\pi}{d}$, $s_r = \frac{1}{4}$. Otherwise, for $r = 1 + \frac{C}{d}$, s_r is the smallest positive solution of

$$C = 8\pi\frac{s}{(1-s)^2}.$$

Remark 5.13. For $0 < C \leq 2\pi$, we have $0 < s_r \leq 3 - \sqrt{8}$. For $C = 1$, we have $s_r \approx 0.0369 > \frac{1}{28}$.

Proof. We will assume, without loss of generality, that the x-axis is aligned along ζ. Let l be the radius of the largest disk centered at ζ on which f is univalent, that is,

$$D_l(\zeta) \subset f^{-1}(D_{\rho_\zeta}(0)) \subset \text{Basin}(\zeta).$$

The Koebe $\frac{1}{4}$-Lemma (Lemma 2.1), implies

$$l \geq \frac{1}{|f'(\zeta)|} \cdot \frac{\rho_\zeta}{4}. \tag{5.2}$$

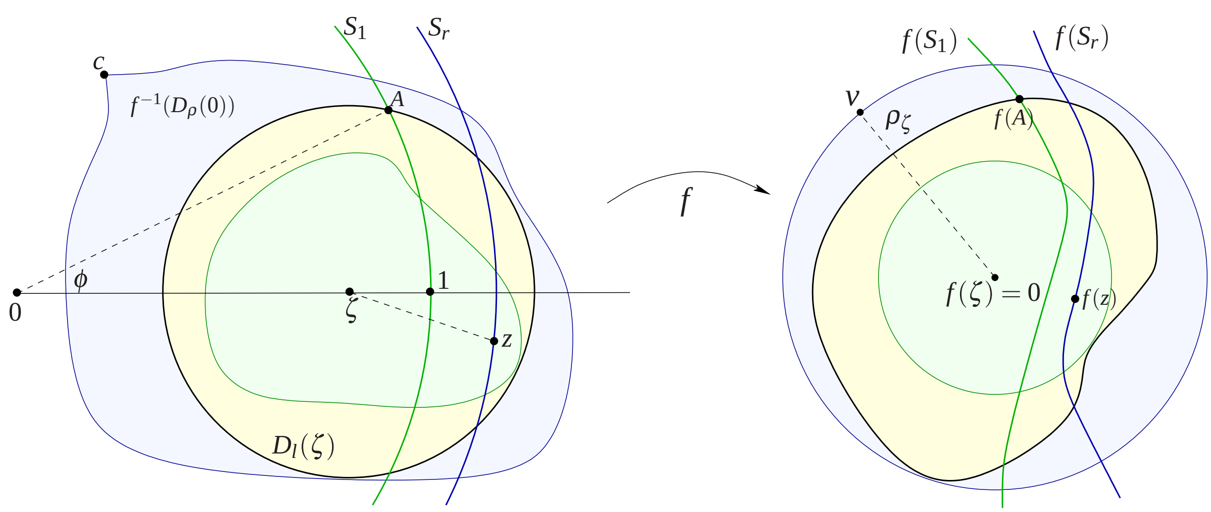

FIGURE 5.1. Using the Koebe Lemma to calculate a lower bound on $|f(z)|$ for z on S_r, in Proposition 5.12.

Let z be a point in Basin(ζ) with $|z| = r$. It is our goal to estimate $|z-\zeta|$. First notice that if $l \leq |z-\zeta|$, the desired result follows immediately from (5.2). Thus, we need only consider the case when $l > |z-\zeta|$.

This means that we can assume that $z \in D_l(\zeta)$, and since $|z| > 1$, there is a point $A \in S_1 \bigcap D_l(\zeta)$; let ϕ be the angle of the sector connecting $0, A$, and 1. See Figure 5.1.

The Koebe Lemma gives an upper bound on $|z-\zeta|$; for $s \leq \frac{|f(z)|}{\rho_\zeta}$,

$$|z-\zeta| \leq \frac{1}{|f'(\zeta)|} \cdot \rho_\zeta \cdot \frac{s}{(1-s)^2}. \tag{5.3}$$

We now look for a lower bound on $|z-\zeta|$ by estimating $\frac{|z-\zeta|}{l}$ for $z \in S_r \cap D_l(\zeta)$. Notice that

$$l = \sqrt{\zeta^2 - 2\zeta\cos(\phi) + 1},$$

since

$$(\cos\phi - \zeta)^2 + \sin^2\phi = l^2$$

where $(\cos(\phi), \sin(\phi))$ is the coordinate of the point A on $S_l(\zeta) \cap S_1$.

From Corollary 5.7, we have

$$\text{Arg}(f(A)) - \text{Arg}(f(\bar{A})) \leq 2\pi,$$

and by Lemma 5.1 (which bounds the radial derivative of f), we have

$$\phi = \text{Arg}(A) \leq \frac{\pi}{d} \cdot \frac{r+1}{r} \leq \frac{2\pi}{d}, \quad \text{for all } r > 1.$$

Since $r = 1 + \frac{C}{d}$, we have

$$\frac{|z-\zeta|}{l} \geq \frac{1 + \frac{C}{d} - \zeta}{\sqrt{\zeta^2 - 2\zeta\cos(\phi) + 1}} \geq \frac{1 + \frac{C}{d} - \zeta}{\sqrt{\zeta^2 - 2\zeta\cos(\frac{2\pi}{d}) + 1}}.$$

Notice that for $0 < C < 2\pi$ and $|\zeta| \leq 1$, the above expression is minimized when $\zeta = 1$. Hence, we have

$$\frac{|z-\zeta|}{l} \geq \frac{\frac{C}{d}}{\sqrt{1 - 2\cos(\frac{2\pi}{d}) + 1}} \geq \frac{C}{2\pi},$$

for all d. This gives us

$$|z-\zeta| \;\geq\; \frac{C\,l}{2\pi} \;\geq\; \frac{C}{2\pi} \cdot \frac{\rho_\zeta}{4|f'\zeta)|}$$

This, together with the upper bound estimate (5.3), gives the lower bound on s as the solution to

$$\frac{C}{2\pi} \cdot \frac{\rho_\zeta}{4|f'(\zeta)|} \leq \frac{s}{(1-s)^2} \frac{\rho_\zeta}{|f'(\zeta)|}$$

This simplifies to

$$C = 8\pi \frac{s}{(1-s)^2},$$

as desired. $\square$

6. THE SIZE OF THE STEP

Each iterate of the algorithm described in § 3 is guided by the values w_n. The difference between w_{n+1} and w_n is called the ***n^{th}-jump*** and is denoted by

$$J_n = A \cdot \left| \frac{f_n}{\alpha_n} \right|,$$

where $f_n = f(z_n)$ and $\alpha_n = \alpha(z_n)$. To be able to control the algorithm we have to carefully adjust the range of the coefficient A. In this section we will explain the choice $A = \frac{1}{15}$.

If f were linear, the algorithm would follow w_n exactly, and $f_n \equiv w_n$. When the degree of f is at least 2, there will be a small error

$$\delta_n = |f_n - w_n|.$$

While the algorithm is described in terms of $\mathbb{C}_{source}$ (the z_n) and $\mathbb{C}_{target}$ ($f(z_n)$ and the w_n), it is more straightforward to think of it in terms of the branched surface $\mathscr{S}$.

Let $r_n \geq 0$ be maximal such that

$$f_{z_0}^{-1} : D_{r_n}(w_n) \to U$$

is univalent, where U is a neighborhood of z_n. This is the distance between $\widehat{w}_n \in \mathscr{S}$ and the critical value $v \in \mathscr{V}_f$ for which $\widehat{w}_n \in \mathrm{Vor}(v)$. Also, let $R_n \geq 0$ be maximal such that

$$f_{z_0}^{-1} : D_{R_n}(f_n) \to V$$

is univalent, where V is a neighborhood of z_n. Note that $\widehat{f_n}$ could be in $\mathrm{Vor}(v')$ for a critical value different from that used for $\widehat{w}_n$; in this case, we still use $R_n = |v' - f_n|$.

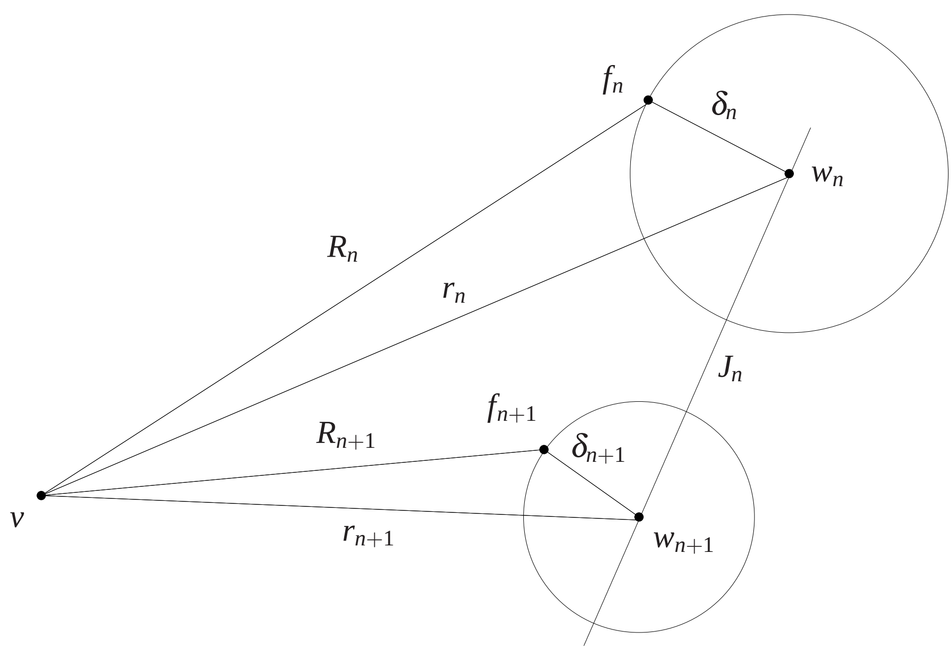

FIGURE 6.1. The various notations used througout this section, shown in the target space.

The following Proposition is a crucial ingredient for the estimate of the average cost.

Proposition 6.1.

$$J_n \geq \frac{1}{66} \cdot r_n.$$

We need to do a little bit of work before proving this proposition. Let

$$\varepsilon_n = z_{n+1} - z_n \qquad \text{and} \qquad h_n = -(z_{n+1} - z_n) \cdot \frac{f'_n}{f_n} = -\varepsilon_n \cdot \frac{f'_n}{f_n}.$$

We use $f'_n = f'(z_n)$, $f''_n = f''(z_n)$, and $f_n^{(j)} = f^{(j)}(z_n)$ as notation for the derivatives of f at z_n.

Lemma 6.2. *If $|\alpha_n h_n| < 1$ then*

$$\delta_{n+1} = |f_{n+1} - w_{n+1}| \leq |h_n f_n| \cdot \frac{|\alpha_n h_n|}{|1 - \alpha_n h_n|}.$$

Proof. Note that since

$$z_{n+1} = z_n - \frac{w_{n+1} - f_n}{f'_n}, \qquad \text{we have} \qquad w_{n+1} = f_n - (z_{n+1} - z_n) f'_n = (1 - h_n) f_n.$$

Thus,

$$\begin{aligned}
\delta_{n+1} = |f_{n+1} - (1-h_n) f_n| &= |f(z_n + \varepsilon_n) - (1-h_n) f_n| \\
&= \left| f_n + f'_n \varepsilon_n + \frac{f''_n}{2!} \varepsilon_n^2 + \cdots - f_n + h_n f_n \right| \\
&= \left| \frac{f''_n}{2!} \varepsilon_n^2 + \frac{f_n^{(3)}}{3!} \varepsilon_n^3 + \dots \right| \\
&= |h_n f_n| \cdot \left| \frac{f''_n}{2!\, f'_n} \varepsilon_n + \frac{f_n^{(3)}}{3!\, f'_n} \varepsilon_n^2 + \dots \right| \\
&\leq |h_n f_n| \cdot \left| \alpha_n \frac{f'_n}{f_n} \varepsilon_n + (\alpha_n \frac{f'_n}{f_n} \varepsilon_n)^2 + \dots \right| \\
&\leq |h_n f_n| \cdot \left| \alpha_n h_n + (\alpha_n h_n)^2 + \dots \right| \\
&\leq |h_n f_n| \cdot \frac{|\alpha_n h_n|}{|1 - \alpha_n h_n|}.
\end{aligned}$$

□

The proof of the following can be found in [BCSS] (Lemma 8.2b and Prop 8.3b). Here $\gamma_n = \gamma(z_n)$ is as in equation (3.2); thus $\alpha_n = \left|\frac{f_n}{f'_n}\right| \gamma_n$.

Lemma 6.3. *Let $u_n = \alpha_n h_n$ and $\psi(u) = 1 - 4u + 2u^2$. Then if $u_n < 1 - 1/\sqrt{2}$, we have*

$$\left| \frac{f'_n}{f'_{n+1}} \right| \leq \frac{(1-u_n)^2}{\psi(u_n)} \qquad \textit{and} \qquad \frac{\gamma_{n+1}}{\gamma_n} \leq \frac{1}{(1-u_n)\psi(u_n)}$$

Remark 6.4. In [BCSS], u_n is defined as $(z_{n+1} - z_n)\gamma_n$. We use

$$h_n = \frac{w_{n+1} - f_n}{f_n} = -(z_{n+1} - z_n)\frac{f_n'}{f_n},$$

and so our usage and that of [BCSS] agree.

The proof of Proposition 6.1 will use induction. Given a choice for the positive numbers A and c we will use the following induction hypothesis

$$\text{Ind}_n(A,c) : \delta_n \leq c \cdot \frac{|f_n'|}{\gamma_n}. \tag{6.1}$$

The constants $A, c > 0$ will be chosen later. The optimization process is better illustrated by using these general parameters instead our final value $A = 1/15$.

Lemma 6.5. *The induction hypothesis* $\text{Ind}_n(A,c)$ *implies*

$$|\alpha_n h_n| \leq A + c.$$

Proof. Observe,

$$\begin{aligned}
|h_n f_n| &= |f_n - w_{n+1}| \\
&\leq |w_n - w_{n+1}| + |f_n - w_n| \\
&\leq J_n + \delta_n \\
&\leq A \cdot \frac{|f_n|}{\alpha_n} + c \cdot \frac{|f_n|}{\alpha_n} = (A+c) \cdot \frac{|f_n|}{\alpha_n}
\end{aligned}$$

□

So that we may apply Lemma 6.3, we impose the condition

$$A + c < 1 - \frac{1}{\sqrt{2}}.$$

By virtue of Lemma 6.5, this condition also ensures that the hypothesis of Lemma 6.2 is satisfied.

Assume $\text{Ind}_n(A,c)$. We will prepare the induction step. From the proof of Lemma 6.5, we have

$$|h_n f_n| \leq (A+c)\frac{|f_n'|}{\gamma_n}.$$

In Lemma 6.2, we obtained

$$\delta_{n+1} \leq \left| h_n f_n \frac{\alpha_n h_n}{1 - \alpha_n h_n} \right| \leq (A+c)\frac{|f_n'|}{\gamma_n} \cdot \frac{u_n}{1-u_n}.$$

Consequently, a sufficient condition which implies $\text{Ind}_{n+1}(A,c)$ is

$$(A+c)\frac{|f_n'|}{\gamma_n} \cdot \frac{u_n}{1-u_n} \leq c \cdot \frac{|f_{n+1}'|}{\gamma_{n+1}},$$

or equivalently,

$$(A+c) \cdot \frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{|f_n'|}{|f_{n+1}'|} \cdot \frac{1}{c} \cdot \frac{u_n}{1-u_n} < 1.$$

From Lemma 6.3, after simplification we get

$$(A+c)\frac{u_n}{\psi(u_n)^2}\cdot\frac{1}{c}<1.$$

Since $u_n \leq A+c$ and $u/\psi(u)$ increases monotonically for $u \in [0, 1-1/\sqrt{2}]$, we must have

$$\frac{(A+c)^2}{\psi(A+c)^2}\cdot\frac{1}{c}<1. \tag{6.2}$$

We have established the following.

Lemma 6.6. *If (A,c) satisfies (6.2) then*

$$Ind_n(A,c) \implies Ind_{n+1}(A,c)$$

The iterations are guided by the points w_n which decrease towards 0 with jumps

$$J_n = A\cdot\left|\frac{f_n}{\alpha_n}\right|.$$

To optimize this convergence we need to find the largest $A>0$ for which there is a $c>0$ such that the pair (A,c) satisfies inequality (6.2). Numerics show that such solutions exist for $A < 0.0703039 < 1/14.22396$; we can use $A = 1/15$ and $c = 0.0158$. Recall,

$$\delta_0 = 0 < 0.0158\cdot\left|\frac{f_0}{\alpha_0}\right|.$$

So $\mathrm{Ind}_0(\frac{1}{15}, 0.0158)$ holds. Then Lemma 6.6 implies that

$$\delta_n \leq 0.0158\cdot\left|\frac{f_n}{\alpha_n}\right| \tag{6.3}$$

holds for all $n \geq 0$.

The proof of Proposition 6.1 uses the following Lemma. This is essentially Corollary 4.3 of [K88]; the lower bound of $\frac{1}{4}$ follows from the Extended Löwner's Theorem in [Sm81].

Lemma 6.7.

$$\frac{1}{4}\cdot R_n \leq \frac{|f_n|}{\alpha_n} \leq (3-2\sqrt{2})\cdot R_n.$$

With these lemmas in hand, we can now return to the proof of Prop. 6.1:

Proof of Proposition 6.1. From Lemma 6.7, we get

$$J_n = A\cdot\frac{|f_n|}{\alpha_n} \geq \frac{|f_n|}{15}\cdot\frac{R_n}{4|f_n|} = \frac{1}{60}\cdot\frac{R_n}{r_n}\cdot r_n.$$

The radius of convergence at w_n is

$$r_n = |w_n - v_n|,$$

where v_n is the critical value for which $\widehat{w}_n \in \mathscr{S}$ lies in $\mathrm{Vor}\big(v_n\big)$. It might be that the radius at f_n is determined by another critical value, say

$$R_n = |f_n - v'_n|.$$

Let $r'_n = |w_n - v'_n|$. Then we have

$$\begin{aligned} r_n \le r'_n &\le |v'_n - f_n| + |f_n - w_n| \\ &= R_n + \delta_n. \end{aligned}$$

In the case when $v_n = v'_n$ we get the same estimate for r_n. Notice, by using (6.3) and Lemma 6.7,

$$\begin{aligned} r_n &\le R_n + \delta_n \\ &\le R_n + 0.0158 \cdot \frac{|f_n|}{\alpha_n} \\ &\le R_n + \frac{0.0158}{3-2\sqrt{2}} R_n \\ &\le 1.09209 \cdot R_n. \end{aligned}$$

Consequently, we have

$$J_n \ge \frac{r_n}{1.09209 \cdot 60} > \frac{r_n}{66},$$

as desired. □

Lemma 6.8. *If* $\alpha_n > 0.1307$*, then*

$$|f_n| \le 1.1376\,|w_n| \qquad \text{and} \qquad |w_{n+1}| \ge 0.41982\,|w_n|.$$

Proof. Observe,

$$|f_n| \le w_n + \delta_n \le |w_n| + 0.0158 \cdot \frac{|f_n|}{\alpha_n}.$$

Hence,

$$|f_n| \le \frac{1}{1 - \frac{0.0158}{\alpha_n}}\,|w_n| \le 1.1376\,|w_n|.$$

Now,

$$\begin{aligned} |w_{n+1}| &= |w_n| - \frac{1}{15} \cdot \frac{|f_n|}{\alpha_n} \\ &\ge |w_n| \cdot \left(1 - \frac{1}{15} \cdot \frac{1}{\alpha_n - 0.0158}\right) \\ &\ge |w_n| \cdot \left(1 - \frac{1}{15} \cdot \frac{1}{0.1307 - 0.0158}\right) \ge 0.41982\,|w_n|. \end{aligned}$$

□

Using these results, we can also obtain a relationship between the guide point w_N where the algorithm terminates and ρ_ζ, the norm of the closest critical value to 0. Recall that $\alpha_N \le 0.1307$ but $\alpha_{N-1} > 0.1307$.

Lemma 6.9. *For $r \geq 1+\frac{1}{d}$*

$$|w_N| \geq \frac{1}{87} \cdot \rho_\zeta.$$

Proof. From Proposition 5.12, we have

$$|w_0| \geq s_r \cdot \rho_\zeta \geq \frac{\rho_\zeta}{28}.$$

If $w_N = w_0$, the lemma holds trivially.

If $N > 0$, then $\alpha_{N-1} \geq 0.1307$ (and $\alpha_N \leq 0.1307$).

From Lemma 6.7, we get

$$\begin{aligned} |f_{N-1}| &\geq \frac{1}{4} \cdot \alpha_{N-1} \cdot R_{N-1} \\ &\geq 0.032675 \cdot R_{N-1} > \frac{R_{N-1}}{31} \\ &\geq \frac{1}{31} \cdot \Big| \rho_\zeta - |f_{N-1}| \Big|. \end{aligned}$$

This last inequality follows from the triangle inequality: if v is the critical value with $|v| = \rho_\zeta$, then 0, v, and f_{N-1} form a triangle with side lengths ρ_ζ, R_{N-1}, and $|f_{N-1}|$. Rewriting the above yields

$$|f_{N-1}| \geq \frac{1}{32} \cdot \rho_\zeta. \tag{6.4}$$

We now apply Lemma 6.8 to obtain

$$|w_N| \geq 0.41982 \cdot |w_{N-1}| \geq 0.41982 \cdot \frac{f_{N-1}}{1.1376} \tag{6.5}$$

The lemma follows by combining equations (6.4) and (6.5). □

7. The Pointwise Cost

In this section we will estimate the number $\#_f(z_0)$ of iterates needed to find an approximate zero starting at z_0. We need some preparation to be able to state the estimate. To simplify notation and without loss of generality, throughout this section we shall assume that ℓ_{w_0} lies along the positive real axis. Furthermore, we shall assume that no critical values of f lie along ℓ_{w_0}.

As before, let $w_0 = f(z_0)$ and the let the w_n be the guide points along ℓ_{w_0} as produced by the algorithm. Also let $\widehat{w}_0 = \widehat{f}(z_0)$ and $\widehat{w}_n$ be the corresponding points in the surface $\mathscr{S}$, lying along the ray $\widehat{\ell}_{w_0}$.

We divide $\widehat{\ell}_{w_0}$ into subintervals as follows: as noted in Proposition 4.4, for each $v \in \mathscr{V}_f$ the intersection of $\widehat{\ell}_{w_0}$ with $\mathrm{Vor}(v)$ will either be an interval or the empty set. Set $\widehat{q}_0 = \widehat{w}_0$, and denote the first interval by $[\![\widehat{q}_0, \widehat{q}_1]\!]$ with corresponding critical value v_1. In general, set

$$[\![\widehat{q}_{j-1}, \widehat{q}_j]\!] = \mathrm{Vor}(v_j) \cap \widehat{\ell}_{w_0}.$$

Let $\beta = \beta(z_0)$ denote the total number of such intervals. Note that for a point $z_0 = re^{2\pi i t_0}$ on our initial circle, we have

$$\beta(z_0) = \mathrm{card}\, \mathscr{I}_{t_0}.$$

So that we may work in the target space $\mathbb{C}$ rather than in the surface $\mathscr{S}$, we make the following observation. The projection π is an isometry in a neighborhood of $\widehat{\ell}_{w_0}$, since $\mathscr{V}_f \cap \widehat{\ell}_{w_0} = \emptyset$. We define a set $U(\widehat{\ell}_{w_0}) \subset \mathscr{S}$ as

$$U(\widehat{\ell}_{w_0}) = \left\{ \widehat{y} \mid [\![\widehat{y}, \widehat{y}_\perp]\!] \neq \emptyset \right\},$$

where for $y \in \mathbb{C}$, $y_\perp$ denotes the orthogonal projection of y onto ℓ_{w_0} (or its extension ℓ_{-w_0}).

That is, for each critical point c_i which influences the orbit of w_0, we remove the ray perpendicular to ℓ_{w_0} starting at the critical value $f(c_i)$. Lifting the result to $\mathscr{S}$ via the branch of π^{-1} taking ℓ_{w_0} to $\widehat{\ell}_{w_0}$ yields the set $U(\widehat{\ell}_{w_0})$.

Observe that π is an isometry on $U(\widehat{\ell}_{w_0})$, and furthermore, $U(\widehat{\ell}_{w_0})$ contains $\widehat{\ell}_{w_0}$ and a unique lift of each of the points f_n produced by the algorithm. Consequently, we have a well-defined correspondence between the target space $\mathbb{C}$ (minus finitely many rays) and a subset of $\mathscr{S}$ most relevant to the α-step algorithm starting at z_0. In what follows, we shall use the notation

$$\mathrm{vor}(v_i) = \pi(\mathrm{Vor}(v_i) \cap U(\widehat{\ell}_{w_0})),$$

and shall slightly abuse notation by using v_i for $f(c_i)$.

Note that the branch of f^{-1} which takes w_0 to z_0 is well-defined throughought all of $\pi(U(\widehat{\ell}_{w_0}))$; in particular, it coincides with analytic continuation of f^{-1} along ℓ_{w_0}.

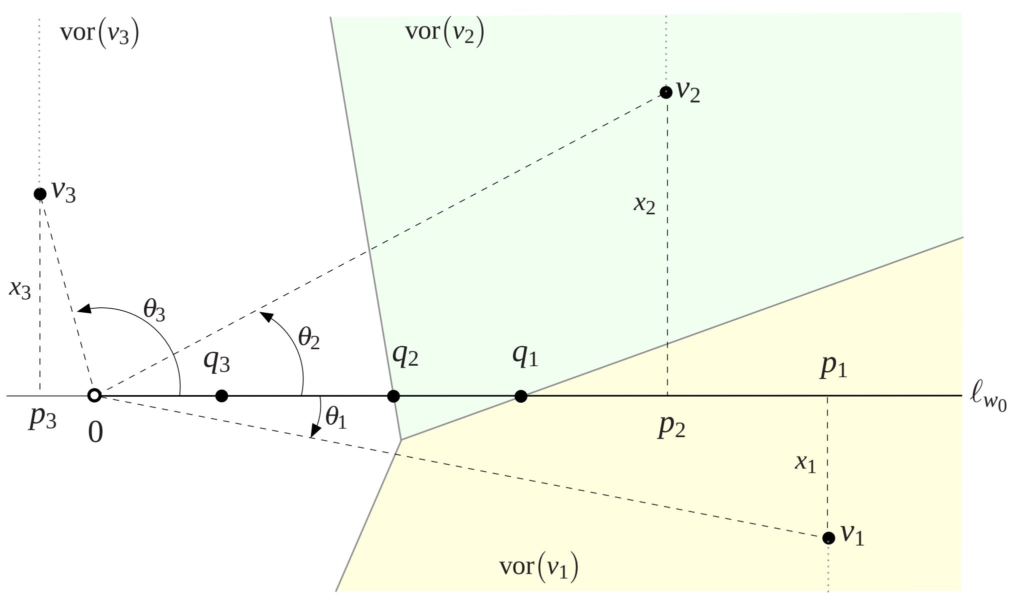

FIGURE 7.1. We divide ℓ_{w_0} into intervals where it is influenced by each critical value; the various notations used in this section are labeled as in the figure.

Let p_j be the orthogonal projection of v_j onto the ray ℓ_{w_0} (or its extension, ℓ_{-w_0}), and let $x_j = |v_j - p_j|$. See Figure 7.1. Also, let $\theta_j \in (-\pi, \pi]$ be the angle between v_j and the ray ℓ_{w_0}; that is,

$$\theta_j = \mathrm{Arg}(v_j / w_0).$$

Furthermore, use $\beta^+(z_0)$ to denote the number of θ_j for which $|\theta_j| \leq \pi/2$ (or, equivalently, for which p_j lies on ℓ_{w_0}).

With this notation in hand, we can state an upper bound on the cost of finding an approximate zero starting from a point z_0.

Proposition 7.1. *Let z_0 be an initial point for the α-step path lifting algorithm, with $|z_0| > 1$, let $f \in \mathscr{P}_d(1)$, $w_0 = f(z_0)$. Then the maximum number of steps required for the algorithm to produce an approximate zero starting from z_0 is*

$$\#_f(z_0) \leq 67 \cdot \left(\log \frac{|w_0|}{|w_N|} + \beta^+(z_0) \log \frac{9}{4} + \sum_{j=1}^{\beta^+(z_0)} \log \left(\frac{4 + \tan|\theta_j|}{\sec|\theta_j| - 1} \right) \right),$$

where $\beta^+(z_0)$ is the number of relevant critical values along ℓ_{w_0} with angle $|\theta_j| < \pi/2$, and w_N is the final “guide-point” for the algorithm.

Remark 7.2. The above result may seem circular, since w_N cannot be determined *a priori*. However, Lemma 6.9 tells us that $\rho_\zeta/87 \leq |w_N| < \rho_\zeta$.

In order to establish this proposition, we estimate the number of steps required to pass each Voronoi domain, and then sum over the $\beta(z_0)$ domains that ℓ_{w_0} passes through.

If w_j and w_k are two guide points lying on ℓ_{w_0} with $k > j$, we can define the rather trivial function $\mathrm{Cost}(w_j, w_k) = k - j$. This measures the number of iterations required by the α-step algorithm beginning at a point z_j near w_j to obtain a point z_k near w_k. We extend this function to all pairs of points y_1 and y_2 lying on ℓ_{w_0} by linear interpolation. It is our goal in this section to estimate $N = \mathrm{Cost}(w_0, w_N)$ where w_N is an approximate zero.

Rather than count the number of steps directly (which is possible, but tedious), instead we follow a suggestion of Mike Shub and integrate the reciprocal of the stepsize along ℓ_{w_0}.

Lemma 7.3. *Let y_1 and y_2 be two points of ℓ_{w_0}. Then*

$$\mathrm{Cost}(y_1, y_2) \leq 67 \int_{y_2}^{y_1} \frac{dy}{r_y},$$

where $r_y = |y - v|$ for each $y \in \mathrm{vor}(v) \cap \ell_{w_0}$.

Proof. Recall that in section 6, we used J_n to denote the n^{th} jump, that is, $J_n = |w_n - w_{n+1}|$ where w_n is a guide point for the algorithm. Set $J(w_n) = J_n$, and extend the function $J(y)$ to all of ℓ_{w_0} by linear interpolation. Now consider the differential equation along ℓ_{w_0} given by

$$\frac{dy}{dt} = -J(y) \qquad y(0) = w_0. \tag{7.1}$$

Since $J(y)$ is Lipschitz, the equation (7.1) has a unique solution. Observe that the points w_n are exactly the values given by using Euler’s method with stepsize 1 to solve (7.1) numerically.

Now consider instead the differential equation given by

$$\frac{dy}{dt} = -\frac{r_y}{67} \qquad y(0) = w_0. \tag{7.2}$$

We wish to compare the solution of (7.2) to the Euler method for (7.1). We will show that for every y in any interval $[w_{n+1}, w_n]$, we have $r_y/67 \leq J(y)$. Consequently, if $\varphi(t)$ is the solution to (7.2) and $\varphi(t_1) = y_1$, $\varphi(t_2) = y_2$, then we will have $t_2 - t_1 \geq \mathrm{Cost}(y_1, y_2)$.

To see that $r_y/67 \leq J_y$ for all $y \in [w_{n+1}, w_n]$, we must examine a few cases. First, note that if $y \in \mathrm{vor}(v_i)$, we have

$$r_y^2 = (y - p_i)^2 + x_i^2.$$

Also, recall that by virtue of Prop. 6.1, we have $J(w_n) \geq r_{w_n}/66$.

First consider the case where the interval $[w_{n+1}, w_n]$ lies entirely in $\mathrm{vor}(v_i)$. If $w_{n+1} \geq p_i$, then since r_y is decreasing on the interval $[p_i, w_n]$, we have $J(y) \geq r_y/66$. If $p_i \geq w_{n+1}$, r_y will be nondecreasing. However, we can apply the triangle inequality (recalling that $J(w_n) = w_n - w_{n+1}$) to see that

$$r_y \leq J(w_n) + r_{w_n} \leq J(w_n) + 66J(w_n),$$

and so $J(w_n) \geq r_y/67$ for all y in the interval.

In the case where the interval intersects more than one Voronoi region, we proceed as follows. First, observe that for all $y \in [q_i, w_n]$, we have already established that $J(y) \geq r_y/67$ holds (where q_i is the smallest point of $[w_{n+1}, w_n] \cap \mathrm{vor}(v_i)$). Since $|v_i - q_i| = |q_i - v_{i+1}|$, we have $J(q_i) \geq r_{q_i}/67$, and we continue as above.

Finally, the equation (7.2) is separable; elementary calculus yields

$$t(y) = 67 \int_y^{w_0} \frac{dy}{r_y}.$$

□

Let y be a point on ℓ_{w_0}, and let c be a critical point which influences w_0; as before, let p be the orthogonal projection of $f(c)$ onto ℓ_{w_0}, and let x denote the distance between $f(c)$ and p.

For each y and a fixed critical point c, we also define the angle A_y, which is the angle that the segment from y to $f(c)$ makes with the segment between $f(c)$ and p. Notice that $r_y = |f(c) - y|$. As before, use θ_c to denote the angle between $f(c)$ and ℓ_{w_0}. See Figure 7.2.

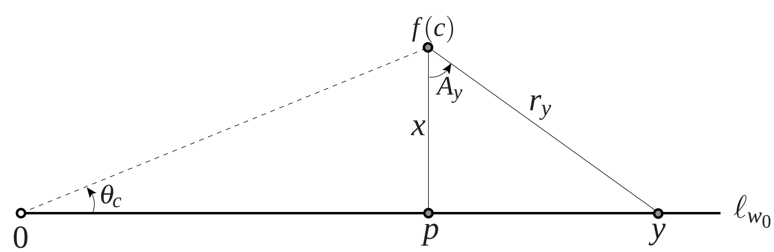

FIGURE 7.2. The quantities y, r_y, p, x, A_y, and θ_c.

We now define the following function, related to $\mathrm{Cost}(y_1, y_2)$.

$$\pounds(y_1, y_2, c) = \log\left(\frac{(y_1 - p) + r_{y_1}}{(y_2 - p) + r_{y_2}}\right).$$

By virtue of Lemma 7.3, if y_1 and y_2 are both in $\mathrm{vor}\big(f(c)\big)$, we have

$$\mathrm{Cost}(y_1, y_2) \leq 67 \int_{y_2}^{y_1} \frac{dy}{r_y} = 67\,£(y_1, y_2, c). \tag{7.3}$$

However, £ will still be useful even when one or both of its first two arguments are not in $\mathrm{vor}\big(f(c)\big)$. We establish some bounds on the value of £ in the next few lemmas.

Lemma 7.4.

$$r_y + (y-p) \leq \begin{cases} 3(y-p) & \textit{if } A_y > \frac{\pi}{6} \\ x\sqrt{3} & \textit{if } A_y \leq \frac{\pi}{6} \end{cases}$$

Proof. Note that $r_y + (y-p) = x(\tan A_y + \sec A_y)$. If $A_y > \pi/6$, we have $x(\tan A_y + \sec A_y) \leq 3x\tan A_y = 3(y-p)$. When $A_y \leq \pi/6$, note that $\tan A_y + \sec A_y$ is increasing in A_y; at $A_y = \pi/6$, $r_y + (y-p) = x\sqrt{3}$.

We remark that this holds even if $p < 0$. □

Lemma 7.5. *Let* $y_1, y_2 \in \ell_{w_0}$ *with* $y_1 > y_2 \geq 3p > 0$. *Then*

$$£(y_1, y_2, c) < \log\frac{y_1}{y_2} + \log\frac{9}{4}.$$

Proof. We consider two cases: when the angle A_y is large and when it is small.

If $A_{y_1} \leq \pi/6$, since $y_2 > p$

$$£(y_1, y_2, c) < £(y_1, p, c) \leq \log\frac{x\sqrt{3}}{x} = \log\sqrt{3},$$

where we have used Lemma 7.4 in the second inequality.

If $A_{y_1} > \pi/6$, we have (using Lemma 7.4 again)

$$£(y_1, y_2, c) \leq \log\frac{3(y_1 - p)}{2(y_2 - p)} = \log\frac{3y_1(1 - p/y_1)}{2y_2(1 - p/y_2)}.$$

Since $y_2 \geq 3p$, we have $(1 - p/y_1)/(1 - p/y_2) < 3/2$, and so

$$£(y_1, y_2, c) \leq \log\frac{y_1}{y_2} + \log\frac{9}{4}.$$

Since $\sqrt{3} < 9/4$, the above bound holds in either case. □

Lemma 7.6. *If* $p > 0$,

$$£(3p, 0, c) \leq \log\frac{4 + \tan|\theta_c|}{\sec|\theta_c| - 1}.$$

We note that since $p > 0$, we have $-\pi/2 < \theta_c < \pi/2$. Consequently, $\frac{4+\tan|\theta_c|}{\sec|\theta_c|-1} > 1$.

Proof. We have

$$£(3p, 0, c) = \log\frac{(3p - p) + r_{3p}}{r_0 - p} \leq \log\frac{2p + (2p + p\tan|\theta_c|)}{p\sec|\theta_c| - p} = \log\frac{4 + \tan|\theta_c|}{\sec|\theta_c| - 1}.$$

□

Finally, we handle the case where $|\theta_c| \geq \pi/2$.

Lemma 7.7. *If* $y_1 > y_2 > 0 \geq p$, $\quad £(y_1, y_2, c) \leq \log(y_1/y_2)$.

Proof. Observe that $r_{y_2} \geq y_2 - p$, since r_{y_2} is the hypotenuse of the right triangle with a leg of length $y_2 - p$. Also, by the triangle inequality, $r_{y_1} - r_{y_2} \leq y_1 - y_2$.

Using this, we have

$$\begin{aligned}
\frac{r_{y_1} + (y_1 - p)}{r_{y_2} + (y_2 - p)} &\leq \frac{(r_{y_2} + y_1 - y_2) + (y_1 - p)}{2(y_2 - p)} \\
&= \frac{2y_1 - p + r_{y_2} - y_2}{2(y_2 - p)} \\
&\leq \frac{2(y_1 - p) + r_{y_2} - (y_2 - p)}{2(y_2 - p)} \\
&\leq \frac{y_1 - p}{y_2 - p} < \frac{y_1}{y_2}.
\end{aligned}$$

Consequently, $£(y_1, y_2, c) = \log \frac{r_{y_1} + (y_1 - p)}{r_{y_2} + (y_2 - p)} < \log(y_1/y_2)$ as desired. □

We can now prove the main result of this section.

Proof of Proposition 7.1. First, divide ℓ_{w_0} into segments where it intersects each of the $\beta(z_0)$ Voronoi regions $\mathrm{vor}(v_j)$; the j^{th} segment will be bounded by points q_{j-1} and q_j (we set $q_0 = w_0$, and $q_{\beta(z_0)} = w_N$). See Figure 7.1.

Now, we have

$$N = \mathrm{Cost}(w_0, w_N) = \sum_{j=1}^{\beta(z_0)} \mathrm{Cost}(q_{j-1}, q_j) \leq 67 \sum_{j=1}^{\beta(z_0)} £(q_{j-1}, q_j, c_j), \tag{7.4}$$

where the inequality follows from Lemma 7.3 and (7.3). Applying Lemmas 7.5 and 7.6 gives us

$$\sum_{j=1}^{\beta^+(z_0)} £(q_{j-1}, q_j, c_j) \leq \sum_{j=1}^{\beta^+(z_0)} \log^+ \left| \frac{q_{j-1}}{q_j^*} \right| + \beta^+(z_0) \log \frac{9}{4} + \sum_{j=1}^{\beta^+(z_0)} \log \frac{4 + \tan|\theta_j|}{\sec|\theta_j| - 1}$$

where $q_j^* = \max(|q_j|, |3p_j|)$.

Note that since $q_j^* \geq |q_j|$, replacing q_j^* with q_j will still give us an upper bound; furthermore, since $|q_{j-1}| > |q_j|$, the logarithm of their ratio is positive. Thus, we have

$$\sum_{j=1}^{\beta^+(z_0)} £(q_{j-1}, q_j, c_j) \leq \sum_{j=1}^{\beta^+(z_0)} \log \left| \frac{q_{j-1}}{q_j} \right| + \beta^+(z_0) \log \frac{9}{4} + \sum_{j=1}^{\beta^+(z_0)} \log \frac{4 + \tan|\theta_j|}{\sec|\theta_j| - 1}. \tag{7.5}$$

Now we apply Lemma 7.7 to the remaining intervals (if any).

$$\sum_{j=\beta^+(z_0)+1}^{\beta(z_0)} £(q_{j-1}, q_j, c_j) \leq \sum_{j=\beta^+(z_0)+1}^{\beta(z_0)} \log \left| \frac{q_{j-1}}{q_j} \right| \tag{7.6}$$

Combining equations (7.5) and (7.6) with (7.4) and recalling that $q_0 = w_0$, $q_\beta = w_N$ gives the desired result. □

8. THE AVERAGE COST

In this section we shall prove our Main Theorem (Thm. 1.1), which follows from averaging the bound in Proposition 7.1 over the starting values on the circle of radius $r = 1 + C/d$.

Recall from Definition 4.6 that $\mathscr{I}$ is the set of pairs (t,c) for which the critical points $c \in \mathscr{C}_f$ influence the starting values $z_0 = re^{it}$ on the initial circle of radius r, $\mathscr{I}_t$ is the set of critical points which influence a given t, and $\mathscr{I}_c$ are the $t \in S_r$ which are influenced by c.

For each pair in $(t,c) \in \mathscr{I}$, we use $\theta = \theta(t,c)$ to denote the angle between $[0, f(re^{2\pi it})]$ and $[0, f(c)]$, that is

$$\theta(t,c) = \operatorname{Arg} \frac{f(re^{2\pi it})}{f(c)}.$$

In the notation of section 7, $\theta(t,c_j) = \theta_j$ where $v_j = \widehat{f}(c_j)$ and $(t,c_j) \in \mathscr{I}$.

Note that for each fixed c, $\mathscr{I}_c$ is a collection of finitely many intervals: $\mathscr{I}_c$ consists of for those t such that $\widehat{\ell}_{f(re^{it})}$ intersects $\operatorname{Vor}\big(\widehat{f}(c)\big)$.

Define for every $c \in \mathscr{C}_f$ the function $\theta_c : \mathscr{I}_c \to \mathbb{R}$ by

$$\theta_c(t) = \theta(t,c) = \operatorname{Arg} \frac{f(re^{2\pi it})}{f(c)}.$$

Lemma 8.1. *For each $c \in \mathscr{C}_f$, the map θ_c is at most (m_c+1)-to-one.*

Proof. For every $\theta \in (-\pi, \pi]$ there are at most (m_c+1) rays $\widehat{\ell} \subset \mathscr{S}$ for which the angle between $[0, f(c)]$ and $\pi(\widehat{\ell})$ is θ and which also intersect $\operatorname{Vor}\big(\widehat{f}(c)\big)$. This is a consequence of Proposition 4.4. □

As an immediate consequence of Lemma 5.1, we have

$$2\pi d \cdot \frac{r}{r+1} \leq \frac{d}{dt}\theta_c(t) \leq 2\pi d \cdot \frac{r}{r-1}. \tag{8.1}$$

Proposition 8.2. *Let $f \in \mathscr{P}_d(1)$ be of degree d and $r > 1$. Then*

$$\int_0^1 \sum_{\substack{c \in \mathscr{I}_t \\ |\theta(t,c)| < \pi/2}} \log \frac{4 + \tan|\theta(t,c)|}{\sec|\theta(t,c)| - 1}\, dt \leq 3 \cdot \frac{r+1}{r}.$$

Proof. Througout the proof, let $\psi(\theta) = \frac{4+\tan|\theta|}{\sec|\theta|-1}$. From Lemma 8.1 and (8.1), we see that for fixed values of c, we have

$$\int\limits_{\substack{t \in \mathscr{I}_c \\ |\theta_c(t)| < \pi/2}} \log \psi(\theta_c(t))\, dt \leq (m_c+1) \int_{-\pi/2}^{\pi/2} \log \psi(\theta) \frac{d\theta}{\theta_c'(t)} \leq (m_c+1) \frac{r+1}{2\pi r d} \int_{-\pi/2}^{\pi/2} \log \psi(\theta)\, d\theta.$$

Thus

$$\begin{aligned}\int_0^1 \sum_{\substack{c\in\mathscr{I}_t\\ |\theta_c(t)|<\pi/2}} \log\psi(\theta(t,c))\,dt &= \sum_{c\in\mathscr{C}_f}\int_{\substack{t\in\mathscr{I}_c\\ |\theta_c(t)|<\pi/2}} \log\psi(\theta(t,c))\,dt\\ &\le \sum_{c\in\mathscr{C}_f}(m_c+1)\frac{r+1}{2\pi rd}\int_{-\pi/2}^{\pi/2}\log\psi(\theta)\,d\theta\\ &\le \frac{2d-2}{2\pi d}\cdot\frac{r+1}{r}\cdot 9.2901\\ &< 3\cdot\frac{r+1}{r}.\end{aligned}$$

□

Recall that $\beta^+(z)$ denotes the number of critical points that influence the orbit of $z = re^{2\pi it}$ with the critical value in the same half-plane, i.e.,

$$\beta^+(re^{2\pi it}) = \operatorname{card}\{c\in\mathscr{I}_t \mid -\pi/2<\theta(t,c)<\pi/2\}.$$

The next proposition bounds the number of such Voronoi domains a starting value encounters, on average.

Proposition 8.3.

$$\int_0^1 \beta^+(re^{2\pi it})dt \le \frac{1+r}{r}.$$

Proof. Note that

$$\int_0^1 \beta^+(re^{2\pi it})\,dt = \int_0^1 \sum_{\substack{c\in\mathscr{I}_t\\ |\theta_c(t)|<\pi/2}} 1\,dt = \sum_{c\in\mathscr{C}_f}\int_{\substack{t\in\mathscr{I}_c\\ |\theta_c(t)<\pi/2}} 1\,dt.$$

As in the proof of Proposition 8.2, we transport the calculation from the source space to the target space using the bound on $\theta_c'(t)$ in (8.1) and the fact that for fixed c, $\theta_c(t)$ is at most (m_c+1)-to-one (Lemma 8.1). This gives us

$$\int_0^1 \beta^+(re^{2\pi it})\,dt \le \sum_{c\in\mathscr{C}_f}\int_{-\pi/2}^{\pi/2}\frac{d\theta}{\theta_c'(t)} \le \sum_{c\in\mathscr{C}_f}(m_c+1)\frac{r+1}{2\pi rd}\cdot\pi \le 2(d-1)\frac{r+1}{2rd} < \frac{r+1}{r}.$$

Above, we used the fact that $\sum_{c\in\mathscr{C}_f} m_c = d-1$. □

Lemma 8.4. *If* $r \ge 1+\frac{1}{d}$

$$\int_0^1 \log\frac{|w_0|}{|w_N|}\,dt \le d\log r + \log 87 + \frac{1}{d}\cdot\frac{1+r}{r}\cdot|\log K_f|.$$

Proof. Proposition 5.9, Proposition 5.12, Lemma 5.1, and Lemma 6.8 are used in the following calculation.

$$\begin{aligned}\int_0^1 \log\frac{|w_0|}{|w_N|}\,dt &= \int_0^1 \log|w_0|\,dt - \int_0^1 \log|w_N|\,dt \\ &\le d\log r - \int_0^1 \log\frac{\rho_\zeta}{87}\,dt \\ &\le d\log r + \log 87 + \sum_{\zeta\in\mathscr{Z}_f} |\log\rho_\zeta|\cdot\frac{1}{d}\cdot\frac{1+r}{r} \\ &\le d\log r + \log 87 + \frac{1}{d}\cdot\frac{1+r}{r}\cdot|\log K_f|\end{aligned}$$

□

Remark 8.5. If $r = 1+\frac{1}{d}$, then $d\log r < 1$, giving $\int_0^1 \log\frac{|w_0|}{|w_N|}\,dt \le 1+\log 87+\frac{2}{d}|\log K_f|$.

Now we are ready to provide a proof of the main theorem.

Proof of Theorem 1.1. Let $r = 1+1/d$. Proposition 7.1, Proposition 8.3, Lemma 8.4, and Proposition 8.2 imply

$$\begin{aligned}\overline{\#_f} &= \int_0^1 \#_f(re^{2\pi it})\,dt \\ &\le \int_0^1 67\cdot\left[\log\frac{|w_0|}{|w_N|} + \beta^+(re^{2\pi it})\log\frac{9}{4} + \sum_{\substack{c\in\mathscr{I}_t \\ |\theta(t,c)|<\pi/2}} \log\frac{4+\tan|\theta(t,c)|}{\sec|\theta(t,c)|-1}\right]dt \\ &\le 67\left[\left(1+\log 87+\frac{2}{d}|\log K_f|\right)+1.622+6\right] \\ &\le 67\cdot\left[13.1+\frac{2}{d}|\log K_f|\right].\end{aligned}$$

□

9. Concluding Remarks

(1) Our goal in this work was to bound the number of iterations of the α-step algorithm, rather than to optimize the arithmetic complexity. Since each step of the algorithm requires computing of all of the derivatives of f, one could use a higher-order method instead of Newton's method (see [K88], [Ho], [SS86]) in the algorithm without a significant increase in cost, setting $z_{n+1} = T_k(f(z_n)-w_{n+1})$, the k^{th} truncation of the Taylor series. Use of such a method would result in a larger stepsize (and consequently fewer steps).

(2) Alternatively, the use of α could be curtailed (or even entirely removed) by dynamically adjusting the guide points w_n as follows. At each step, set w_{n+1} to be $(1-h_n)|f(z_n)|w$. Initially, take $h_n = h_0$, but if $f(z_n)$ is not sufficiently close to w_{n+1}, divide h_n by 2 and try again until it is. At the next step, start with $h_{n+1} = \min(h_0, 2h_n)$. Note that this approach,

while similar in spirit, is somewhat different from the variable stepsize methods explored in [HS]. One can still use α to detect whether an approximate zero has been located, or, if evaluating higher derivatives of f is impractical, other methods such as those in [B02] or [O] can be used.

(3) The ideas used in this paper can be adapted to those used in [KS] to locate approximate zeros for all d of the roots of f. That work uses a path-lifting method, but initial points are taken much further outside the unit circle, and a fixed stepsize is taken. In order to take the initial points on the circle of radius $1+1/d$, one needs to apply Lemma 5.1 to ensure that the initial points are properly spaced in the target space.

(4) Using some of the ideas in [GLSY], one should be able to extend these results to deal with multiple roots or root clusters, for which K_f becomes arbitrarily large.

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